Published by Institute of Physics Publishing for SISSA/ISAS

Received: *February 17, 2004*
Revised: *April 13, 2004*
Accepted: *May 3, 2004*

E_{11} and spheric vacuum solutions of eleven- and ten dimensional supergravity theories

Igor Schnakenburg

Racah Institute of Physics, The Hebrew University
Jerusalem 91904, Israel
E-mail: igorsc@phys.huji.ac.il

André Miemiec

Institut für Physik, Humboldt Universität
D-12489 Berlin, Newtonstr. 15, Germany
E-mail: miemiec@physik.hu-berlin.de

Abstract: In view of the newly conjectured Kac-Moody symmetries of supergravity theories placed in eleven and ten dimensions, relations between these symmetry groups and possible compactifications are examined. In particular, we identify the relevant group cosets that parametrise the vacuum solutions of $AdS \times S$ type.

Keywords: Global Symmetries, Supergravity Models.

Contents

1. Introduction

Searching for solutions of higher dimensional supergravity theories is an extensive program. The numerous solutions that have been found so far preserve different amounts of supersymmetry, or break supersymmetry altogether. In this paper we are mainly concerned with sphere reductions of 11d supergravity and IIB supergravity constructed in the spirit of [1]. These solutions, together with torus compactifications preserve a large amount of supersymmetry. The consistency of the corresponding Kaluza-Klein truncations was analyzed in [2]. It was noted that for a consistent truncation to exist, there must occur an extension of the isometry algebra by the gauge field [3]. It is therefore desirable to formulate (super)gravity theories in a way that makes their symmetry structure easily accessible. This idea was pursued in [4], where a nonlinear description of the gauge fields was advocated. After dimensional reduction the degrees of freedom which derive from the graviton turn into scalars or gauge potentials, and thus can also be casted in a description via nonlinear realisations.

However, gravity can also be taken into account in a nonlinear realisation *before* compactification, as has been outlined in [5]–[14]. This approach suggests a class of infinite dimensional groups to play a role in the formulation of supergravity (superstring) theories. The generating algebras are Kac-Moody algebras (KMA). The relation between these algebras and uncompactified (super) gravity theories has first been suggested in [5]. Afterwards it was shown that almost all (super)gravities admit a description in terms of non-linear realisation of groups generated by subsets of the generators of a KMAs [6]–[9]. The low level roots of these algebras were shown to be in one-to-one correspondence with the field contents of these theories in [10]. Further properties of these particular Kac-Moody algebras were presented in [11]. Relations to string theory were examined in [12, 13].

The question we focus on here is to identify those subsets of generators that generate the isometries of vacuum solutions of the supergravity under consideration. We will show how some known symmetry algebras occur from E_{11}. In the next two sections we want to discuss the situation in eleven and ten dimensions.

JHEP05(2004)003

2. Possible vacua of $D = 11$ supergravity

The nonlinear realisations under consideration are cosets G/H where G is generated by a KMA and H is generated by all those generators that are left invariant under a modified Cartan involution, called 'temporal' involution. It was presented in [12] and defined in the following way. The algebra $\mathfrak{g}$ of G can be decomposed into a sum $\mathfrak{g} = \Delta^- \oplus \text{CSA} \oplus \Delta^+$, where $\Delta^\pm$ are called positive/negative roots and CSA denotes the Cartan subalgebra. On the elements of the Cartan subalgebra one defines an action by

$$H_a \to -H_a \,, \tag{2.1}$$

and on the simple positive and negative roots by

$$E_\alpha \leftrightarrow -\epsilon_\alpha E_{-\alpha} \,, \tag{2.2}$$

where α is parametrising any set of simple positive roots, and ϵ_α is defined according to the number of indices of the relevant generator that is pointing in the time-like direction, i.e. it is taken to be +1 if the number of indices pointing in the time direction ('1') is even, and -1 otherwise. The action on the simple roots extends to an action on all roots. The linear combination $E_\alpha - \epsilon_\alpha E_{-\alpha}$ is left invariant under this involution. The coset G/H is obviously parametrised by the linear combinations $E_\alpha + \epsilon_\alpha E_{-\alpha}$ and the elements of the Cartan subalgebra.

An important example are A_{d-1} algebras with generators $K^a{}_b$. The linear combination $E_\alpha - \epsilon_\alpha E_{-\alpha}$ singles out the antisymmetric generators $J_{ab} = K_{ab} - K_{ba}$. Here the indices are pulled down with the flat metric η_{ab}, which we have chosen to be $\eta_{ab} = \text{diag}(-, + \ldots +)$. These are the generators of the Lorentz group $\text{SO}(1, d-1)$.[1]

In the non-linear realisation G/H we take a group element $g \in G$ to transform as

$$g \to g_0 \, g \, h^{-1} \,, \tag{2.3}$$

with $h \in H$. If the Cartan form $g^{-1}dg$ is enhanced by the Lorentz connection term ω [5]–[14]

$$\mathcal{V} = g^{-1}dg - \omega \,, \tag{2.4}$$

then $\mathcal{V}$ transforms covariantly under H

$$\mathcal{V} \to h \mathcal{V} h^{-1} \,. \tag{2.5}$$

In particular, if H is the Lorentz group (see above) then the corresponding Cartan form $\mathcal{V}$ is Lorentz covariant [14].

In [5] it was conjectured that eleven dimensional supergravity admits a description as a nonlinear realisation of the Kac-Moody algebra E_{11}, whose Dynkin diagram is depicted in figure 1.

[1] The 'pure' Cartan involution leads to the invariant subgroup $\text{SO}(d)$.

JHEP05(2004)003

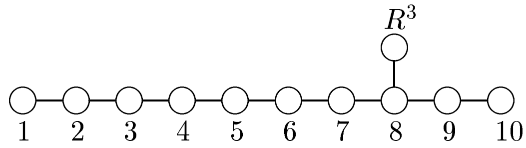

Figure 1.

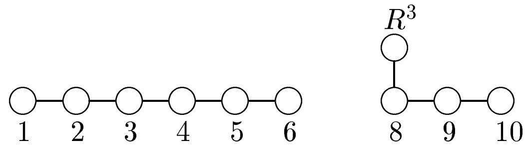

Figure 2.

This algebra obviously contains the subalgebra A_{10}, which in turn contains the subalgebra $\mathfrak{so}(1,10)$. It is the corresponding Lorentz group $\mathrm{SO}(1,10)$, that becomes local (H) in the nonlinear realisation G/H. The particular Lorentz signature is a first consequence of the modified Cartan involution.

It is convenient to split representations of E_{11} into representations of A_{10} since in the latter it is simple to find the Lorentz subalgebra. This analysis of splitting representations with respect to their A_n subalgebra has been carried out in [9, 10] and we briefly give the results. According to the number the simple root $R^{9\,10\,11}$ appears in the non-simple roots of E_{11} the representations of A_{10} are organised in levels. At level zero of the decomposition the adjoint representation of $A_{10} \sim \mathrm{SL}(11)$ is reproduced, while at level 1, 2, and 3 the generators

$$R^{a_1a_2a_3}\,, \qquad R^{a_1\ldots a_6}\,, \qquad R^{a_1\ldots a_8,b} \qquad a_1,\ldots a_8, b = 1\ldots 11 \tag{2.6}$$

occur, respectively.

We next want to split the A_{10} subalgebra into subalgebras A_6 and A_3 by deleting the fourth node of the A_{10} subalgebra from the right. This corresponds to a split of $\mathrm{SL}(11)$ into $\mathrm{SL}(4)$ and $\mathrm{SL}(7)$ where both subspaces completely decouple

$$[K^{\hat{a}}{}_{\hat{b}},\, K^i{}_j] \equiv 0\,, \tag{2.7}$$

(the unhatted $K^i{}_j$ lives in seven dimensions and the hatted one in four). The reason for these special values will become obvious in due course. We note that it is consistent with the commutation relations of E_{11} to couple the level 1 generator to the left side of the subalgebra A_3, see figure 2. $A_4 \oplus A_6$ is a proper, finite dimensional, and decomposable subalgebra of E_{11}.

This includes that the 3-form generator can merely take values in the four dimensional subspace A_3 it couples to. We check explicitly that the 3-form potential enhances the aforementioned $\mathrm{SL}(4)$ to $\mathrm{SL}(5)$ by taking the commutation relations of $R^{\hat{a}_1\hat{a}_2\hat{a}_3}$ with $K^{\hat{a}}{}_{\hat{b}}$ written in terms of new generator introduced by:

$$K^0{}_{\hat{c}} = \frac{1}{3!}\epsilon_{\hat{a}_1\hat{a}_2\hat{a}_3\hat{c}}R^{\hat{a}_1\hat{a}_2\hat{a}_3}\,, \qquad K^{\hat{c}}{}_0 = \frac{1}{3!}\epsilon^{\hat{a}_1\hat{a}_2\hat{a}_3\hat{c}}R_{\hat{a}_1\hat{a}_2\hat{a}_3}\,. \tag{2.8}$$

We find

$$[K^{\hat a}{}_{\hat b},\, K^0{}_{\hat c}] = -\delta^{\hat a}_{\hat c}\, K^0{}_{\hat b} + \delta^{\hat a}_{\hat b}\, K^0{}_{\hat c}\,, \qquad [K^{\hat a}{}_{\hat b},\, K^{\hat c}{}_0] = \delta^{\hat c}_{\hat b}\, K^{\hat a}{}_0 - \delta^{\hat a}_{\hat b}\, K^{\hat c}{}_0\,. \tag{2.9}$$

Taking out a trace part by $\hat K^{\hat a}{}_{\hat b} = K^{\hat a}{}_{\hat b} - \frac{1}{3}\,\delta^{\hat a}_{\hat b}\, \sum_{\hat c=8}^{11} K^{\hat c}{}_{\hat c}$ we obtain

$$[\hat K^{\hat a}{}_{\hat b},\, K^0{}_{\hat c}] = -\delta^{\hat a}_{\hat c}\, K^0{}_{\hat b}\,, \qquad [\hat K^{\hat a}{}_{\hat b},\, K^{\hat c}{}_0] = -\delta^{\hat c}_{\hat b}\, K^{\hat a}{}_0\,, \tag{2.10}$$

which provide the correct commutation relations to form the algebra of SL(5) if we also add the new diagonal element

$$K^0{}_0 = \frac{1}{3}\sum_{\hat a=8}^{11} K^{\hat a}{}_{\hat a}\,. \tag{2.11}$$

The typical shape of the Dynkin diagram of SL(5) can be obtained from the simple, positive roots (labels as on the right hand side of figure 2)

$$E_{R^3} = K^0{}_8, \qquad E_8 = K^8{}_9\,, \qquad E_9 = K^9{}_{10}\,, \qquad E_{10} = K^{10}{}_{11}\,, \tag{2.12}$$

and the elements of the CSA given by

$$H_{R^3} = -2K^8{}_8 - K^9{}_9 - K^{10}{}_{10} - K^{11}{}_{11}\,, \quad H_8 = K^8{}_8 - K^9{}_9 - 2K^{10}{}_{10} - 2K^{11}{}_{11}\,, \tag{2.13}$$

$$H_9 = K^9{}_9 - K^{10}{}_{10} - 2K^{11}{}_{11}\,, \qquad H_{10} = K^{10}{}_{10} - K^{11}{}_{11}\,. \tag{2.14}$$

In the full non-linear realisation of E_{11} one finds the covariant quantity belonging to the generator $R^{\hat a_1 \hat a_2 \hat a_3}$ by calculating the Cartan form and taking the closure with the conformal group [6, 7, 14]. The resulting object is $F_{\hat a_1 \hat a_2 \hat a_3 \hat a_4} = 4\partial_{[\hat a_1} A_{\hat a_2 \hat a_3 \hat a_4]}$ and can be identified with the field strength of eleven dimensional supergravity. Since the splitting of the algebra allows dependence on 4 dimensions only, we have to conclude that this field strength is proportional to $\epsilon_{\hat a_1 \hat a_2 \hat a_3 \hat a_4}$ (Freund-Rubin ansatz).

The method of nonlinear realisations applied to supergravity theories requires doubling of the fields, i.e. apart from the 3-form generator which generates the 3-form gauge potential, one also introduces the magnetic dual in form of a 6-form generator belonging to a 6-form gauge potential. The Lorentz covariant field equation that reduces the doubled degrees of freedom is a generalised self-duality relation

$$F^{a_1 a_2 a_3 a_4} = \frac{1}{7!}\epsilon^{a_1 \ldots a_{11}} F_{a_5 \ldots a_{11}}\,. \tag{2.15}$$

This equation indicates that the field strength of the 6-form gauge potential is living in the space orthogonal to the chosen 4 dimensions. To this 6-form potential belongs the generator $R^{c_1 \cdots c_6}$. We are forced to conclude, that the 6-form generator which is redundant in the Dynkin diagram of E_{11} since it does not correspond to a simple root must be included by hand due to the non-trivial equation of motion and the requirement that it does not vanish. This poses an algebraic puzzle since there is no root of E_{11} which might enlarge the A_6 subalgebra, while keeping the SL(5) algebra unaffected.[2] However, introducing

$$K^{-1}{}_i = \frac{1}{6!}\epsilon_{i i_1 \ldots i_6} R^{i_1 \ldots i_6}\,, \tag{2.16}$$

$$K^i{}_{-1} = \frac{1}{6!}\epsilon^{i i_1 \ldots i_6} R_{i_1 \ldots i_6}\,, \quad i = 1, \ldots, 7 \tag{2.17}$$

[2]We thank A.Kleinschmidt for discussion on that subject.

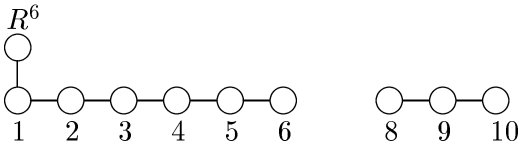

Figure 3.

and using the commutation relations of E_{11} on this seven dimensional subspace (indices $1, \ldots, 7$), we find analogous formulas to (2.8)–(2.10). This suggests coupling of the 6-form generator to the seven-dimensional subspace. In fact, there is a root λ of E_{11} which is orthogonal at least to the A_3 subalgebra (labels 8, 9, 10) and leads to an extension of A_6 to A_7. It reads:

$$\lambda = \alpha_2 + 2\alpha_3 + 3\alpha_4 + 4\alpha_5 + 5\alpha_6 + 6\alpha_7 + 6\alpha_8 + 4\alpha_9 + 2\alpha_{10} + 2\alpha_{11} \tag{2.18}$$

and is depicted in figure 3.

The simultaneous presence of gauge generators R^3 and R^6 (as required by the equation of motion) both of which enhance the relevant A_3 and A_6 subalgebras of A_{10} can only be consistent with the algebraic structure of E_{11} if commutation relations between elements of either algebra effectively vanish. This blindfolding of E_{11} indeed occurs due to the field which is associated with the generator $R^{8,1}$. As shown, the vanishing of "mixing terms" is trivial for the generators $\hat{K}^{\hat{a}}{}_{\hat{b}}$ and $K^i{}_j$, see (2.7). The only non-trivial "mixing" commutator is

$$[R^{\hat{a}_1\hat{a}_2\hat{a}_3},\, R^{i_1\ldots i_6}] \sim_{E_{11}} R^{i_1\ldots i_6[\hat{a}_1\hat{a}_2,\hat{a}_3]}\,, \tag{2.19}$$

which is non-vanishing in E_{11} (and therefore $A_4 \oplus A_7$ is not a subalgebra of E_{11}). The generator on the right hand side is of course the level 3 generator from equation (2.6), antisymmetric in eight indices. We thus realise that the corresponding field cannot exist without destroying the split of SL(11) caused by deleting the 7th node as we have done above. So we are provided with a natural cut-off for the roots of E_{11}. One can use the Jacobi identity to show that all higher roots of E_{11} cannot be present after this split since

$$[R^6,\, R^6] = [R^6,\, [R^3_1,\, R^3_2]] = [[R^6,\, R^3_1],\, R^3_2] + [R^3_1,\, [R^6,\, R^3_2]] \equiv 0\,. \tag{2.20}$$

This argument shows that one can introduce the notion of an "effective subalgebra" SL(8)$\oplus$ SL(5) in E_{11}, where "effective" means that although the generator $R^{8,1}$ is present in the algebra of E_{11} the corresponding field in the gravity theory cannot be present without spoiling the $11 \to 7 + 4$ dimensional split. Thus in the gravitational theory the fields corresponding to generators of level higher than 2 have to be realised trivially. If the level 3 generator is realised trivially[3] then SL(8) $\oplus$ SL(5) becomes a subalgebra. This process is depicted in figure 4. Of course, this $A_7 \oplus A_4$ algebra is finite.

In the following we concentrate on the local subalgebras that arise via the temporal involution as was indicated in the opening of this paragraph. Since the E_{11} algebra is assumed to be given in its maximal non-compact form, so is its $A_4 \sim$ SL(5) subalgebra.

[3]This generator tends to be a latent source for troubles since its interpretations as a gravity dual is not fully understood.

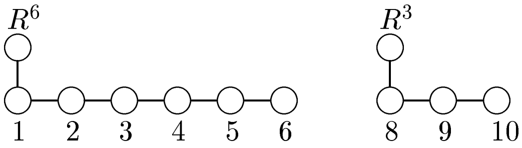

Figure 4.

We thus identify SO(5) as the compact subalgebra which is taken to be local in the relevant coset SL(5)/ SO(5). This coset is parametrised by the fields of the theory which only live in four dimensions spanned by the A_3 subalgebra of A_{10}, i.e. by the (traceless) generators $K_{\hat{a}\hat{b}} + K_{\hat{b}\hat{a}}$ and the linear combination $R^{\hat{a}_1\hat{a}_2\hat{a}_3} + R_{\hat{a}_1\hat{a}_2\hat{a}_3}$ as well as the elements of the Cartan subalgebra given by appropriate linear combinations of $K^{\hat{a}}{}_{\hat{a}}$ ($\hat{a} = 8, \dots, 11$). The local subgroup is spanned by the generators

$$K_{\hat{a}\hat{b}} - K_{\hat{b}\hat{a}} \quad \text{and} \quad R^{\hat{a}_1\hat{a}_2\hat{a}_3} - R_{\hat{a}_1\hat{a}_2\hat{a}_3}\,, \tag{2.21}$$

with minus signs since none of the indices is time-like. It has been noted in [15,12,7] that a generator coupling to the far left side of an A_{d-1} subalgebra has the same Lorentz transformation properties as the momentum generator. This is precisely the situation we encountered in eq. (2.8). In particular we have to apply the temporal involution to the generator $\hat{P}_{\hat{a}}$. Still, since none of the indices is time-like we find the local subalgebra to be spanned by generators[4]

$$J_{\hat{a}\hat{b}} = K_{\hat{a}\hat{b}} - K_{\hat{b}\hat{a}}, \quad \text{and} \quad P_{\hat{c}} = \hat{P}_{\hat{c}} - \eta_{\hat{c}\hat{d}}\hat{P}^{\hat{d}}\,, \tag{2.22}$$

where indices are shoveled about with $\eta_{\hat{a}\hat{b}} = (+,+,+,+)$. The generators $J_{\hat{a}\hat{b}}$ correspond to the usual Lorentz generators in the 4 dimensional subalgebra of $A_3 \sim \mathrm{SL}(4)$. In view of the last equation (2.22) one can further identify the momentum generator with a generator of the Lorentz algebra in a new direction #

$$P_{\hat{a}} \equiv J_{\hat{a}\,\#}\,. \tag{2.23}$$

This momentum generator can now be taken to parametrise the coset $\frac{\mathrm{SO}(5)}{\mathrm{SO}(4)}$, describing the previously discussed reduction of the aforementioned local SO(5) symmetry to SO(4) (pure Lorentz symmetry). This coset is well-known to define the 4-sphere S^4 and indicates that E_{11} naturally contains the part of the vacuum solution that is given by the 4-sphere.

This is not the whole discussion since we have to deal with the left hand side of the diagram in figure 4. Again the node generating the 6-form potential behaves exactly like a momentum operator of this subspace. In particular, the generator corresponding to the simple positive root on the far left of A_7 can in seven dimensions be dualised

$$\hat{P}_1 = \epsilon_{1234567} R^{234567}\,, \tag{2.24}$$

and obviously contains the time-like coordinate. Using the temporal involution on the $A_7 \sim \mathrm{SL}(8)$ algebra and using the momentum generator to enhance the SO(1,6) Lorentz

[4] P_a denotes a generator, and is not the same as $\eta_{ab}P^b$.

JHEP05(2004)003

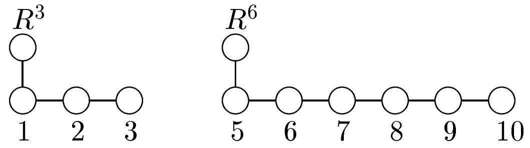

Figure 5.

algebra we find the full local subgroup to be SO(2, 6). This time the generator

$$K_c = \hat{P}_c + \eta_{cd}\hat{P}^d \qquad \eta_{ab} = (-,-,+,+,+,+,+,+) \tag{2.25}$$

parametrises the coset $\frac{\mathrm{SO}(2,6)}{\mathrm{SO}(1,6)}$ which is well-known to be AdS_7. Using terminology of the conformal group we called this 'momentum generator' K_a - the generator of special conformal transformations.

As a complete solution for the vacuum we find $S^4 \times AdS_7$. This is a known result; this time, however, derived from a purely group theoretical point of view.

2.1 Splitting SL(11) into different SL(d) subalgebras

First of all we note that one can also consistently decompose $\mathrm{SL}(11) \to \mathrm{SL}(4) \times \mathrm{SL}(7)$ with the latter tensor product exchanged relative to the previous discussion. This corresponds to the deletion of node 4 in figure 1. The roots of E_{11} generating the **165** representation (R^3) and the **462** representation (R^6) of SL(11) can again be used to extend the obvious $\mathrm{SL}(4) \times \mathrm{SL}(7)$ subalgebra to $\mathrm{SL}(5) \times \mathrm{SL}(7)$ and $\mathrm{SL}(4) \times \mathrm{SL}(8)$ (but not simultaneously!). They are given respectively by:

$$\mathrm{SL}(5) \times \mathrm{SL}(7) : \lambda = \alpha_2 + 2\alpha_3 + 3\alpha_4 + 3\alpha_5 + 3\alpha_6 + 3\alpha_7 + 3\alpha_8 + 2\alpha_9 + \alpha_{10} + \alpha_{11} \tag{2.26}$$
$$\mathrm{SL}(4) \times \mathrm{SL}(8) : \lambda = \alpha_6 + 2\alpha_7 + 3\alpha_8 + 2\alpha_9 + \alpha_{10} + 2\alpha_{11}\,. \tag{2.27}$$

Again, we notice that the field generated by the level 3 generator $R^{a_1\cdots a_8,b}$ can not survive on either subspace. Therefore the split Dynkin diagram shown in figure 5 gives an effective algebra, and we have used the generators $R^{a_1a_2a_3}$, and $R^{a_1\cdots a_6}$ to extend the relevant subalgebras exactly as discussed in the previous section.

Again we can make the identification of the momentum generator with the generator of the 6-form potential. This time the relevant dualisation

$$P_5 = \epsilon_{5\,6\,7\,8\,9\,10\,11} R^{6\,7\,8\,9\,10\,11} \tag{2.28}$$

does not involve a time index (as opposing (2.24)). The temporal involution just generates the subgroup SO(8) which we would like to interpret as SO(7) and the momentum generator. This momentum generator then parametrises the seven sphere S^7. The group covariant expression for 6-form generators (also group covariant with respect to the relevant conformal group) depends only on seven indices $5, \ldots, 11$ and therefore has to be proportional to the totally antisymmetric tensor in this dimension. It is natural to identify the field strength with the volume form of the seven sphere S^7.

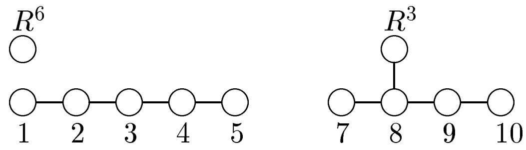

Figure 6.

On the left hand side of the Dynkin diagram we find the 3-form potential to couple to the first node which involves the time direction. The interpretation of the simple positive root as a momentum generator via

$$\hat{P}_1 = \epsilon_{2341} R^{234} \tag{2.29}$$

implies that the invariant generators of the temporal involution span the group SO(2, 3). However, as usually we interpret the additional node as a momentum generator parametrising the coset SO(2, 3)/ SO(1, 3) which corresponds to AdS_4. This solution thereby corresponds to the other known vacuum solution of eleven dimensional supergravity $AdS_4 \times S^7$.

Other cases to consider concern the splitting of $A_{10} \to A_5 \oplus A_4$, depicted in figure 6, where we have already enhanced the relevant subalgebras by the representations antisymmetric in three and six indices respectively, again making use of the fact, that the level 3 generator $R^{a_1 \cdots a_8, b}$ can not couple to either side of the split.

We find $R^{a_1 \cdots a_6}$ not to be coupled to the 6 dimensional subspace to the left either. The extra scalar node (called R_6) can be seen to arrive by using the ϵ symbol of 6d space-time (A_5) to contract all indices of $R^{a_1 \cdots a_6}$.

This time we do not have available the interpretation of momentum generators which can only be used when the additional node couples to the far left side of an A_{d-1} subalgebra. In fact, we do not find it on either side of this split. Another simple thought shows that the Lorentz covariant and gauge invariant object of the theory -the field strengths- for example $F_{a_1 \dots a_7}$ has to vanish since the indices only range over 6 (or 5) different values. The eleven dimensional generalised self-duality condition (which is the only Lorentz covariant way of relating the group covariant objects belonging to dual generators) then automatically puts to zero the dual field strength $\partial_\mu \hat{A}_{a_1 a_2 a_3}$. We find that although the 6-form potential can live in a six dimensional subspace, its field strengths can not.

In a very similar way also the discussion for the splitting $A_{10} \to A_2 \oplus A_7$ goes wrong since one can not interpret the gauge field nodes as 'momentum' generators. Also, the relevant field strengths with antisymmetric indices are not compatible with the dimensionality of the subspaces. The situation becomes even worse if one tries to place the 6-form potential in this A_2 subspace. Finally, it is easy to see that also $A_{10} \to A_1 \oplus A_8$ can be excluded.

3. Other supergravity theories

If the above analysis reproduces the known results in case of eleven dimensional supergravity it is natural to ask whether it also does for the ten dimensional theories. We start

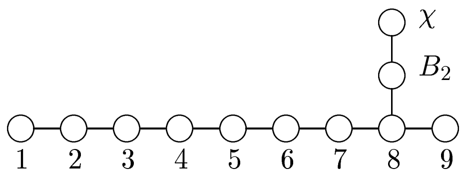

Figure 7.

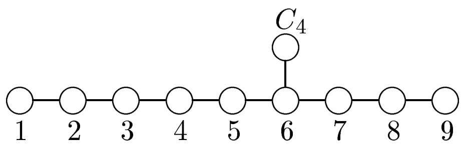

Figure 8.

considering the case of IIB theory. This theory was shown to also be in relation with a non-linear realisation of E_{11} [6]. It might therefore be surprising that we will indeed find the known $S^5 \times AdS_5$ solution for this case as well.

The novel feature here is that the gravity subalgebra is A_9 instead of the full A_{10}. It is natural to decompose the representations of E_{11} into those of A_9 in the way that is indicated in figure 7, where we have placed the two expansion nodes above the relevant gravity line.

The decomposition of representation has in great detail been carried out in [9, 10], where it has also been shown that very-extended E_8, also called E_8^{+++} contains the subalgebra E_7^{+++}. Both algebras contain the same A_9 subalgebra but differ in the remaining nodes. We depict E_7^{+++} in the following figure 8.

As indicated in figure 8 the node apart from the gravity line corresponds to a 4-form potential. In fact, splitting representation of E_7^{+++} into representation of A_9 yields the fields

$$R^{a_1 \dots a_4}, \quad \text{and} \quad R^{a_1 \dots a_7, b} \tag{3.1}$$

at level 1 and 2 respectively. It can be shown that the relevant non-linear realisation describes the theory which is a consistent truncation of IIB supergravity, namely the one where all gauge fields have been set to zero except the self-dual 5-form field strength [6, 10]. However, we are now in the position to resume the discussion from chapter 2. We split the A_9 subalgebra (for good reason) into $A_4 \oplus A_4$ by deleting the middle node $\mathrm{SL}(10) \to \mathrm{SL}(5) \times \mathrm{SL}(5)$. The 4-form generator that normally takes values in all 10 dimensions can now be split into two 4-forms one of which takes values in dimensions $1, \dots, 5$ while the other takes values in dimensions $6, \dots, 10$. We call these generators R_4^+ and R_4^-, where R^+ is the simple root of E_7^{+++} while R_4^- corresponds to the root

$$\alpha_{C_4} + \alpha_9 + 2\alpha_8 + 3\alpha_7 + 4\alpha_6 + 4\alpha_5 + 3\alpha_4 + 2\alpha_3 + \alpha_2 \, . \tag{3.2}$$

These roots can now be used to enhance the relevant A_4 subalgebras in the way indicated by figure 9. The algebra depicted in this figure is not a subalgebra of E_7^{+++}. However, it is an effective algebra describing the 10 dimensional theory since the field generated by the level 2 generator in (3.1) does not respect the 5,5 split of ten dimensions and can in this

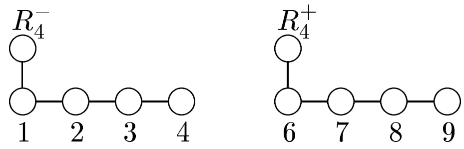

Figure 9.

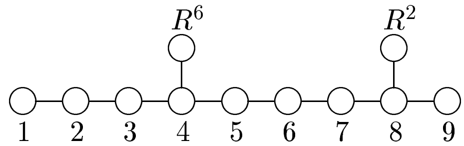

Figure 10.

particular case not be physical. As usually, we interpret that to provide a natural cut-off for *all* higher level generators as they would have to contain this non-physical field in a non-linear realisation.

By this way of drawing, the potentials obviously depend only on the space-time coordinates they are attached to. As in chapter 2 we find it possible to interpret the nodes corresponding to $R_4^{\pm}$ as 'momentum' generators. In applying exactly the same analysis as in the previous chapter we find these momentum generators to parametrise the coset spaces

$$\frac{\mathrm{SO}(2,4)}{\mathrm{SO}(1,4)} \times \frac{\mathrm{SO}(6)}{\mathrm{SO}(5)} \sim AdS_5 \times S^5 \,. \tag{3.3}$$

We merely note that splitting the A_9 subalgebra in any other way will not lead to consistent results.

Performing a similar analysis on other very-extended groups $\mathcal{G}^{+++}$ gives analogous results. Let us briefly carry out the analysis for groups of the D_n-series of Lie algebras, i.e. we take the Kac-Moody groups D_n^{+++}. The discussion goes through for each value of n, but we will draw the diagram of D_8^{+++} which was also considered in [10].

At level 1, 2, and 3 one finds representations of antisymmetric tensors in 2, 6 and 8 indices, as well as a generators with 8 indices of which only 7 are antisymmetric and which we call $R^{7,1}$. This representation content shows that we find enhanced subalgebras of A_9 if we split SL(10) $\to$ SL(7) $\times$ SL(3) (by deleting node 7) or SL(10) $\to$ SL(3) $\times$ SL(7) (by deleting node 3). Repeating the same arguments as in chapter 2, and using the roots

$$\lambda_1 = \alpha_{R^2} \,, \tag{3.4}$$
$$\lambda_2 = \alpha_{R^6} + \alpha_9 + 2\alpha_8 + 3\alpha_7 + 3\alpha_6 + 3\alpha_5 + 3\alpha_4 + 2\alpha_3 + \alpha_2 \,, \tag{3.5}$$

we find evidence for the existence of vacuum solutions $AdS_7 \times S^3$. Using the roots

$$\lambda_1 = \alpha_{R^6} \,, \tag{3.6}$$
$$\lambda_2 = \alpha_{R^2} + \alpha_9 + 2\alpha_8 + 2\alpha_7 + 2\alpha_6 + 2\alpha_5 + 2\alpha_4 + 2\alpha_3 + \alpha_2 \,, \tag{3.7}$$

we also find the $AdS_3 \times S^7$ solution. All other splits of A_9 will not result in appropriate cosets.

In a similar fashion one shows that there does not appear to exist such a vacuum solution for E_6^{+++}. This concludes the discussion for simply laced Kac-Moody algebras. It is easy to check that the only other possible $AdS \times S$ spectrum seems to occur for G_2^{+++} with vacuum solutions $AdS_3 \times S^2$ and $AdS_2 \times S^3$.

4. Conclusions

We consider isometries of vacuum solutions of supergravity theories and identify their algebras within a special class of Kac-Moody algebras, namely the so-called very-extended Lie algebras. The low level generators of these algebras were shown to describe (super)gravity theories in a non-linear realisation. We discover the appropriate isometry algebras after solving a consistency puzzle. One cannot derive the isometry algebras of these vacuum solutions canonically when starting from any other than the very-extended algebras. It is an essential feature of our discussion that the generators corresponding to duals of gravity naturally vanish. By using the temporal involution [12] we furthermore found the right signatures of the subgroups to turn up in the correct way. It would be interesting to discover less simple solutions of 11d supergravity via a similar approach.

Amazingly all the solutions found so far support supersymmetry. In [16] the relation of coset spaces to supersymmetry was explained and it was shown that the Killing spinor equation can be understood geometrically. It would be challenging to see how supersymmetry can be embedded into the full Kac-Moody algebra E_{11}.

Acknowledgments

I.S. is grateful for talks with D. Kazhdan, F. Englert, and P. West. The work of A.M. is supported by the European Commission RTN programme HPRN-CT-2000-00131 and the Deutsche Forschungsgemeinschaft (DFG). The work of I.S. is partly supported by BSF — American-Israel Bi-National Science Foundation, the Israel Academy of Sciences — Centers of Excellence Program, the German-Israel Bi-National Science Foundation, and the European RTN-network HPRN-CT-2000-00122.

References

[1] P.G.O. Freund and M.A. Rubin, *Dynamics of dimensional reduction*, *Phys. Lett.* **B 97** (1980) 233.

[2] B. de Wit and H. Nicolai, *The consistency of the S^7 truncation in $D = 11$ supergravity*, *Nucl. Phys.* **B 281** (1987) 211;
H. Nastase, D. Vaman and P. van Nieuwenhuizen, *Consistent nonlinear kk reduction of 11D supergravity on $AdS_7 \times S^4$ and self-duality in odd dimensions*, *Phys. Lett.* **B 469** (1999) 96 [hep-th/9905075]; *Consistency of the $AdS_7 \times S^4$ reduction and the origin of self-duality in odd dimensions*, *Nucl. Phys.* **B 581** (2000) 179 [hep-th/9911238].

[3] M. Cvetič, H. Lu and C.N. Pope, *Consistent Kaluza-Klein sphere reductions*, *Phys. Rev.* **D 62** (2000) 064028 [hep-th/0003286].

JHEP05(2004)003

[4] E. Cremmer, B. Julia, H. Lu and C.N. Pope, *Dualisation of dualities, I*, *Nucl. Phys.* **B 523** (1998) 73 [hep-th/9710119]; *Dualisation of dualities, II. Twisted self-duality of doubled fields and superdualities*, *Nucl. Phys.* **B 535** (1998) 242 [hep-th/9806106].

[5] P. West, E_{11} *and M-theory*, *Class. and Quant. Grav.* **18** (2001) 3143 [hep-th/0104081].

[6] I. Schnakenburg and P.C. West, *Kac-Moody symmetries of IIB supergravity*, *Phys. Lett.* **B 517** (2001) 421 [hep-th/0107181].

[7] I. Schnakenburg and P.C. West, *Massive IIA supergravity as a non-linear realisation*, *Phys. Lett.* **B 540** (2002) 137 [hep-th/0204207].

[8] N.D. Lambert and P.C. West, *Coset symmetries in dimensionally reduced bosonic string theory*, *Nucl. Phys.* **B 615** (2001) 117 [hep-th/0107209].

[9] P. West, *Very extended* E_8 *and* A_8 *at low levels, gravity and supergravity*, *Class. and Quant. Grav.* **20** (2003) 2393 [hep-th/0212291].

[10] A. Kleinschmidt, I. Schnakenburg and P. West, *Very-extended Kac-Moody algebras and their interpretation at low levels*, *Class. and Quant. Grav.* **21** (2004) 2493 [hep-th/0309198].

[11] M.R. Gaberdiel, D.I. Olive and P.C. West, *A class of lorentzian Kac-Moody algebras*, *Nucl. Phys.* **B 645** (2002) 403 [hep-th/0205068].

[12] F. Englert and L. Houart, *G+++ invariant formulation of gravity and M-theories: exact BPS solutions*, *J. High Energy Phys.* **01** (2004) 002 [hep-th/0311255].

[13] F. Englert, L. Houart and P. West, *Intersection rules, dynamics and symmetries*, *J. High Energy Phys.* **08** (2003) 025 [hep-th/0307024].

[14] P.C. West, *Hidden superconformal symmetry in M-theory*, *J. High Energy Phys.* **08** (2000) 007 [hep-th/0005270].

[15] P. West, $E(11)$, SL(32) *and central charges*, *Phys. Lett.* **B 575** (2003) 333 [hep-th/0307098].

[16] N. Alonso-Alberca, E. Lozano-Tellechea and T. Ortin, *Geometric construction of Killing spinors and supersymmetry algebras in homogeneous spacetimes*, *Class. and Quant. Grav.* **19** (2002) 6009 [hep-th/0208158].